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THE REGULAR FREE-ENDPOINT LINEAR QUADRATIC PROBLEM WITH INDEFINITE COST*

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Abstract. This paper studies an open problem in the context of linear quadratic optimal control, the free-endpoint regular linear quadratic problem with *indefinite* cost functional. It is shown that the optimal cost for this problem is given by a particular solution of the algebraic Riccati equation. This solution is characterized in terms of the geometry on the lattice of all real symmetric solutions of the algebraic Riccati equation as developed by Willems [*IEEE Trans. Automat. Control*, 16 (1971), pp. 621-634] and Coppel [*Bull. Austral. Math. Soc.*, 10 (1974), pp. 377-401]. A necessary and sufficient condition is established for the existence of optimal controls. This condition is stated in terms of a subspace inclusion involving the extremal solutions of the algebraic Riccati equation. The optimal controls are shown to be generated by a feedback control law. Finally, the results obtained are compared with "classical" results on the linear quadratic regulator problem.

Key words. linear quadratic optimal control, indefinite cost functional, free-endpoint problem

AMS(MOS) subject classifications. 93C05, 93C35, 93C60

1. Introduction. In this paper we are concerned with regular, infinite-horizon linear quadratic optimal control problems in which the cost functional is the integral of an *indefinite* quadratic form.

In most of the existing literature on the regular linear quadratic (LQ) problem, it is explicitly assumed that the quadratic form in the cost functional, apart from being positive definite in the control variable alone, is positive semidefinite in the control and state variables simultaneously. In fact, under this semidefiniteness assumption the LQ problem has become quite standard and is treated in many basic textbooks in the field of systems and control [1], [2], [9], [21]. Often a distinction is made between two versions of the problem, the *fixed-endpoint* version and the *free-endpoint* version. In the fixed-endpoint version it is necessary to minimize the cost functional under the constraint that the optimal state trajectory should converge to zero as time tends to infinity, while in the free-endpoint version it is only necessary to minimize the cost functional. For the case that the quadratic form in the cost functional is positive semidefinite both versions of the regular LQ problem are well-understood and completely satisfactory solutions of these problems are available.

Surprisingly, however, for the most general formulation of the regular LQ problem, that is, the case that the quadratic form in the cost functional is indefinite, a satisfactory treatment does not yet exist. In this case we can again distinguish between the fixed-endpoint version and the free-endpoint version. While for the fixed-endpoint version a complete solution has been described in [17] (see also [14]), the free-endpoint version has only been considered in [17] under a very restrictive assumption. Thus we see that, up to now, the free-endpoint regular LQ problem with indefinite cost functional has been an open problem. In the present paper we shall fill up this gap and present a fairly complete solution to this problem.

It is well known [12], [19] that for the free-endpoint regular LQ problem with positive semidefinite cost functional, the optimal cost is given by the smallest positive semidefinite real symmetric solution of the algebraic Riccati equation. We will see that this statement is no longer valid in general if the cost functional is the integral of an

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indefinite quadratic form. It will be shown, however, that in this case also the optimal cost is given by a solution of the algebraic Riccati equation. This particular solution will be characterized in terms of the geometry on the set of all real symmetric solutions of the algebraic Riccati equation as described in [17] and [4].

Another well-known fact is that, for the free-endpoint regular LQ problem with positive semidefinite cost functional, the *existence* of optimal controls is never an issue: under the assumption that the underlying system is controllable, for this problem unique optimal controls always exist for all initial conditions. This is in contrast with the fixed-endpoint LQ problem, where the existence of optimal controls for all initial conditions depends on the "gap" of the algebraic Riccati equation (i.e., the difference between the largest and smallest solutions of the Riccati equation). In this paper we will see that also, for the free-endpoint regular LQ problem with *indefinite* cost functional, optimal controls no longer need to exist for all initial conditions! Moreover, we will establish a necessary and sufficient condition in terms of the "gap" of the algebraic Riccati equation for the existence of optimal controls for all initial conditions. We will show that for the particular case that the cost functional is positive semidefinite this condition is always satisfied, thus explaining the fact that in this special case optimal controls always exist. Finally, we will show that also in the indefinite case the optimal controls for the free-endpoint regular LQ problem, if they exist, are given by a feedback control law.

The outline of this paper is as follows. In the remainder of this section we will introduce most of the notational conventions that will be used. In § 2 we give formulations of both the free-endpoint and fixed-endpoint regular LQ problems that we shall be dealing with. In § 3 we will briefly recall the most important facts that we need on the geometry of the set of all real symmetric solutions to the algebraic Riccati equation as developed in [17] and [4]. In § 4 we will state the solution to the fixed endpoint regular LQ problem with indefinite cost as established in [17]. Also, we will state its (incomplete) counterpart, the solution to the free-endpoint regular LQ problem with positive semidefinite cost functional. Then in § 5 we will state and prove our main theorem, a solution to the free-endpoint regular LQ problem. In order to establish a proof of this theorem we will state and prove a series of smaller lemmas. In § 6 we will show how the "classical" results on the free-endpoint regular LQ problem with positive semidefinite cost functional can be reobtained as a special case of our general solution. We will close this paper in § 7 with some concluding remarks.

We use the following notational conventions. For a given $n \times n$ matrix A its set of eigenvalues will be denoted by $\sigma(A)$. If V is a subspace of $\mathbb{R}^n$ and A is an $n \times n$ matrix then $A|V$ will denote the restriction of A to V. V will be called A-invariant if $AV \subset V$. In this case $\sigma(A|V)$ will denote the set of eigenvalues of $A|V$ and $\sigma(A|\mathbb{R}^n/V)$ will denote the set of eigenvalues of the mapping induced by A in the factor space $\mathbb{R}^n/V$ (see [21]). We will denote subsets of $\mathbb{C}$ by $\mathbb{C}^- := \{s \in \mathbb{C} | \operatorname{Re} s = 0\}$, $\mathbb{C}^0 := \{s \in \mathbb{C} | \operatorname{Re} s = 0\}$, and $\mathbb{C}^+ := \{s \in \mathbb{C} | \operatorname{Re} s > 0\}$. Given a real monic polynomial p there is a unique factorization $p = p_- \cdot p_0 \cdot p_+$ into real monic polynomials with p_-, p_0, and p_+ having all roots in $\mathbb{C}^-$, $\mathbb{C}^0$, and $\mathbb{C}^+$, respectively. If A is a real $n \times n$ matrix and if p denotes its characteristic polynomial then we define $X^-(A) := \ker p_-(A)$, $X^0(A) := \ker p_0(A)$, and $X^+(A) := \ker p_+(A)$. These subspaces are A-invariant and the restriction of A to $X^-(A)(X^0(A), X^+(A))$ has characteristic polynomial $p_-(p_0, p_+)$.

A subset $\mathbb{C}_g$ of $\mathbb{C}$ will be called symmetric if $a + bi \in \mathbb{C}_g \Leftrightarrow a - bi \in \mathbb{C}_g$. If $\mathbb{C}_g$ is given then we define $\mathbb{C}_b := \mathbb{C} \backslash \mathbb{C}_g$. If A is a real $n \times n$ matrix and if p is its characteristic polynomial then, again, p can be factored uniquely into $p = p_g \cdot p_b$, where p_g and p_b are real monic polynomials with all roots in $\mathbb{C}_g$ and $\mathbb{C}_b$, respectively. We denote

$X_g(A) := \ker p_g(A)$ and $X_b(A) := \ker p_b(A)$. Again these subspaces are A-invariant and the restriction of A to $X_g(A)(X_b(A))$ has characteristic polynomial $p_g(p_b)$. In fact, the subspace $X_g(A)(X_b(A))$ is equal to the linear span of all generalized eigenvectors of A corresponding to its eigenvalues in $\mathbb{C}_g(\mathbb{C}_b)$. Alternatively, $X_g(A)(X_b(A))$ is equal to the largest A-invariant subspace V of $\mathbb{R}^n$ such that $\sigma(A|V) \subset \mathbb{C}_g(\mathbb{C}_b)$.

If, in addition to A, a real $p \times n$ matrix C is given, then we denote

$$\langle \ker C|A\rangle := \bigcap_{i=1}^{n} \ker CA^{i-1},$$

the unobservable subspace of (C, A) [21, § 3.2]. Given a symmetric subset $\mathbb{C}_g$ of $\mathbb{C}$ we denote

$$X_{\det} := \langle \ker C|A\rangle \cap X_b(A),$$

the undetectable subspace of (C, A) with respect to $\mathbb{C}_g$. The pair (C, A) is called detectable with respect to $\mathbb{C}_g$ if A is $(\mathbb{C}_{g-})$ stable on the unobservable subspace of (C, A), i.e., if

$$\langle \ker C|A\rangle \subset X_g(A)$$

(see [21, § 3.6]). It is easy to see that (C, A) is detectable if and only if $X_{\det} = 0$. Also, (C, A) is detectable if and only if for all $\lambda \in \mathbb{C}_b$ we have $\ker(A - \lambda I) \cap \ker C = 0$ (see [15]).

In order to be rigorous on the interpretation of the cost functionals that will be considered in this paper, we will now explain what we mean by the statement that the limit of a function *exists in* $\mathbb{R}^e$. Let $\mathbb{R}^e := \mathbb{R} \cup \{-\infty, +\infty\}$. Given $f: \mathbb{R} \to \mathbb{R}$ we say that $\lim_{t\to\infty} f(t)$ exists if it is equal to a real number in the usual sense. We say that $\lim_{t\to\infty} f(t) = -\infty(+\infty)$ if for all $r \in \mathbb{R}$ there exists $T \in \mathbb{R}$ such that $t \geqq T$ implies $f(t) \leqq r(\geqq r)$. Then we say that $\lim_{t\to\infty} f(t)$ exists in $\mathbb{R}^e$ if it exists, is equal to $-\infty$, or is equal to $+\infty$.

If M is a real $n \times n$ matrix and V is a subspace of $\mathbb{R}^n$, then we define $M^{-1}V := \{x \in \mathbb{R}^n | Mx \in V\}$. If V is a subspace of $\mathbb{R}^n$ then $V^\perp$ denotes its orthogonal complement with respect to the standard Euclidean inner product.

Finally, we will denote by $L_{2,\mathrm{loc}}(\mathbb{R}^+)$ the space of all measurable vector-valued functions on $\mathbb{R}^+$ that are square integrable over all finite intervals in $\mathbb{R}^+$. $L_2(\mathbb{R}^+)$ denotes the space of all measurable vector-valued functions on $\mathbb{R}^+$ that are square integrable over $\mathbb{R}^+$. Finally, $L_\infty(\mathbb{R}^+)$ denotes the space of all measurable vector-valued functions on $\mathbb{R}^+$ that are essentially bounded on $\mathbb{R}^+$. Here, $\mathbb{R}^+ := \{t \in \mathbb{R} | t \geqq 0\}$.

2. The regular LQ-problem. Consider the finite-dimensional linear time-invariant system

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0. \tag{2.1}$$

Here, x and u are assumed to take their values in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. A and B are real $n \times n$ and $n \times m$ matrices, respectively. It will be a standing assumption that (A, B) is controllable. We shall consider optimization problems of the type

$$\inf \int_0^\infty \omega(x, u)\, dt, \tag{2.2}$$

where $\omega(x, u)$ is a real quadratic form on $\mathbb{R}^n \times \mathbb{R}^m$ defined by $\omega(x, u) := u^TRu + 2u^TSx + x^TQx$. Here R, S, and Q are assumed to be real matrices such that $R = R^T$ and $Q = Q^T$. As in [17], no a priori definiteness conditions are imposed on the form ω. For a given control function $u \in L_{2,\mathrm{loc}}(\mathbb{R}^+)$, let $x(x_0, u)$ denote the state trajectory of (2.1) and if $T \geqq 0$ let

$$J_T(x_0, u) := \int_0^T \omega(x(x_0, u)(t), u(t))\, dt.$$

We now explain how (2.2) should be interpreted. First, we specify two classes of control functions with respect to which the infimization in (2.2) should be performed. Define

$$U(x_0) := \{u \in L_{2,\mathrm{loc}}(\mathbb{R}^+) \mid \lim_{T\to\infty} J_T(x_0, u) \text{ exists in } \mathbb{R}^e\},$$

$$U_s(x_0) := \{u \in U(x_0) \mid \lim_{t\to\infty} x(x_0, u)(t) = 0\}.$$

Note that, due to the assumption that (A, B) is controllable, we have $U(x_0) \neq \varnothing$ and $U_s(x_0) \neq \varnothing$ for all $x_0 \in \mathbb{R}^n$. For $u \in U(x_0)$ we define

$$J(x_0, u) := \lim_{T\to\infty} J_T(x_0, u). \tag{2.3}$$

We note that $J(x_0, u) \in \mathbb{R}^e$. Now, define

$$V_f^+(x_0) := \inf\{J(x_0, u) \mid u \in U(x_0)\}, \tag{2.4a}$$

$$V^+(x_0) := \inf\{J(x_0, u) \mid \, \mid u \in U_s(x_0)\}, \tag{2.4b}$$

the optimal cost for the free-endpoint problem and fixed-endpoint problem, respectively. By the fact that (A, B) is controllable we have that $V_f^+(x_0)$, $V^+(x_0) \in \mathbb{R} \cup \{-\infty\}$ for all $x_0 \in \mathbb{R}^n$. Following [17], we want to exclude the situation that for certain initial conditions x_0 the values (2.4a) or (2.4b) become equal to $-\infty$. It can be shown that a necessary condition for $V_f^+(x_0) > -\infty$ and $V^+(x_0) > -\infty$ for all x_0 to hold is that $R \geqq 0$ (see [17], [12]). In this paper a standing assumption will be that $R > 0$. Under this assumption the LQ problems defined by (2.4) are called *regular.*

The fixed-endpoint regular LQ problem, defined by (2.4b), was completely resolved in [17] (see also [14]). There, a satisfactory characterization was given for the optimal cost, necessary and sufficient conditions were given for the existence of optimal controls for all initial conditions, and these optimal controls were given in the form of a state-feedback control law. The problems of how to calculate the optimal cost for the free-endpoint regular LQ problem (2.4a), to state necessary and sufficient conditions for the existence of optimal controls, and to calculate these optimal controls have up to now been open. In this paper we will consider these problems.

3. Geometry of the algebraic Riccati equation. A central role in infinite horizon regular linear quadratic control problems is played by the algebraic Riccati equation (ARE)

$$A^TK + KA + Q - (KB + S^T)R^{-1}(B^TK + S) = 0. \tag{3.1}$$

Let Γ denote the set of all real symmetric solutions of the ARE. It was shown in [17] that if Γ is nonempty then it contains a unique element K^+ and a unique element K^- such that

$$\sigma(A - BR^{-1}(B^TK^+ + S)) \subset \mathbb{C}^- \cup \mathbb{C}^0,$$

$$\sigma(A - BR^{-1}(B^TK^- + S)) \subset \mathbb{C}^+ \cup \mathbb{C}^0.$$

Moreover, K^+ and K^- have the additional property that they are the *extremal solutions* of the ARE in the sense that if $K \in \Gamma$ then $K^- \leqq K \leqq K^+$.

Let $\Delta := K^+ - K^-$. Denote $A - BR^{-1}(B^TK^+ + S)$ and $A - BR^{-1}(B^TK^- + S)$ by A^+ and A^-, respectively. If $K \in \Gamma$ define $A_K := A - BR^{-1}(B^TK + S)$. Note that $X^+(A^+) = 0$ and $X^-(A^-) = 0$. Let Ω denote the set of all A^--invariant subspaces contained in $X^+(A^-)$. The following basic theorem is a generalization by Coppel [4] of a theorem that was originally proven by Willems in [17] (see also [16], [10]).

THEOREM 3.1. *Let (A, B) be controllable, and assume that Γ is nonempty. If V is an A^--invariant subspace of $X^+(A^-)$ (that is, if $V \in \Omega$) then $\mathbb{R}^n = V \oplus \Delta^{-1} V^\perp$. There exists a bijection $\gamma : \Omega \to \Gamma$ defined by*

$$\gamma(V) := K^- P_V + K^+(I - P_V),$$

where P_V is the projector onto V along $\Delta^{-1}V^\perp$. If $K = \gamma(V)$ then

$$X^+(A_K) = V,$$

$$X^0(A_K) = \ker \Delta,$$

$$X^-(A_K) = X^-(A^+) \cap \Delta^{-1} V^\perp.$$

Among other things, the result above states that there exists a one-to-one correspondence between the set of all real symmetric solutions of the ARE and the set of all A^--invariant subspaces of $X^+(A^-)$. Following [3], if $K = \gamma(V)$ then we will say that *the solution K is supported by the subspace V.* The next theorem from [4] states that this one-to-one correspondence in fact respects the partial orderings on the sets Γ and Ω.

THEOREM 3.2. *Let (A, B) be controllable and assume that Γ is nonempty. Let K_1 and K_2 be solutions to the* ARE *supported by V_1 and V_2, respectively. Then $K_1 \leqq K_2$ if and only if $V_2 \subset V_1$.*

From the above it follows, for example, that K^- is supported by $X^+(A^-)$ and that K^+ is supported by 0.

4. Classical results. In the present section we briefly summarize the solution of the fixed-endpoint regular LQ problem with indefinite cost functional as outlined in [17]. Subsequently, we will state the well-known result on the free-endpoint regular LQ problem with *positive semidefinite* cost functional. Finally, we will discuss some of the difficulties that can be expected in trying to generalize the latter result to the case that the semidefiniteness assumption is dropped.

Consider the infimization of (2.3) over the class of inputs $U_s(x_0)$. For a given x_0 an input u^* is called *optimal* if $u^* \in U_s(x_0)$ and $J(x_0, u^*) = V^+(x_0)$. The following was proven in [17].

THEOREM 4.1. *Let (A, B) be controllable and assume that $R > 0$. Then we have the following*:

(i) *$V^+(x_0)$ is finite for all $x_0 \in \mathbb{R}^n$ if and only if the* ARE *has a real symmetric solution (i.e., $\Gamma \neq \varnothing$).*

(ii) *If $\Gamma \neq \varnothing$ then for all $x_0 \in \mathbb{R}^n$, $V^+(x_0) = x_0^T K^+ x_0$.*

(iii) *If $\Gamma \neq \varnothing$ then for all $x_0 \in \mathbb{R}^n$ there exists an optimal input u^* if and only if $\Delta > 0$.*

(iv) *If $\Gamma \neq \varnothing$ and $\Delta > 0$ then for each $x_0 \in \mathbb{R}^n$ there is exactly one optimal input u^* and, moreover, this input u^* is given by the feedback control law $u^* = -R^{-1}(B^TK^+ + S)x$.*

As already mentioned, an analogue of the latter theorem for the free-endpoint case, up to now, has only been available for the case that the quadratic form $\omega(x, u)$ is positive semidefinite, i.e., for the case that $\omega(x, u) \geqq 0$ for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$. In the sequel, let $\Gamma_+ := \{K \in \Gamma | K \geqq 0\}$. It is well known [8], [12] that if $\omega \geqq 0$ and if (A, B) is controllable, then the ARE has a smallest positive semidefinite real symmetric solution. More precisely, there exists a (unique) $\tilde{K}$ such that

$$\tilde{K} \in \Gamma_+, \tag{4.1}$$

$$K \in \Gamma_+ \Rightarrow \tilde{K} \leqq K. \tag{4.2}$$

The solution $\tilde{K}$ characterized by (4.1) and (4.2) plays the central role in the solution of the free-endpoint regular LQ problem with positive semidefinite cost. In the following, for a given $x_0 \in \mathbb{R}^n$ an input u^* is called *optimal* if $u^* \in U(x_0)$ and $J(x_0, u^*) = V_f^+(x_0)$.

THEOREM 4.2. *Assume that (A, B) is controllable, that $R > 0$, and that $\omega(x, u) \geqq 0$ for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$. Then we have the following:*

(i) *For all $x_0 \in \mathbb{R}^n$, $V_f^+(x_0) = x_0^T \tilde{K} x_0$.*

(ii) *For each $x_0 \in \mathbb{R}^n$, there is exactly one optimal input u^*, and moreover, this input u^* is given by the feedback control law $u^* = -R^{-1}(B^T \tilde{K} + S)x$.*

Proof. This follows, for example, by combining [12, Thm. 8] and the results from [1, p. 36] (see also [19]). □

We note that in this theorem the *existence* of optimal controls is no issue. In contrast with the fixed-endpoint problem, the positive semidefiniteness assumption assures that in the free-endpoint problem for every initial condition there exists an optimal control.

In trying to generalize the latter theorem to the case that ω is an arbitrary indefinite quadratic form in (x, u) (with of course, as usual, $R > 0$), the following aspects should be considered. First, due to the indefiniteness of ω, the optimal cost $V_f^+(x_0)$ no longer needs to be finite. In this paper we want to restrict ourselves to the case that $V_f^+(x_0)$ is finite for all x_0. In order to establish a condition assuring this, we state the following well-known result. For $v \in \mathbb{R}^m$, denote $\|v\|_R^2 := v^T R v$.

LEMMA 4.3. *Let $K \in \Gamma$. Then for all $u \in L_{2,\mathrm{loc}}(\mathbb{R}^+)$ and for all $T \geqq 0$, we have*

$$J_T(x_0, u) = \int_0^T \|u(t) + R^{-1}(B^T K + S)x(t)\|_R^2 \, dt + x_0^T K x_0 - x^T(T) K x(T).$$

Here, we have denoted $x(t) := x(x_0, u)(t)$.

Proof. For a proof, refer to [2] or [17]. □

In the sequel, let $\Gamma_- := \{K \in \Gamma \mid K \leqq 0\}$. From the previous lemma the following is immediate.

LEMMA 4.4. *Let (A, B) be controllable and $R > 0$. If $\Gamma_- \neq \varnothing$ then $V_f^+(x_0)$ is finite for all $x_0 \in \mathbb{R}^n$.*

Proof. $\Gamma_- \neq \varnothing$ implies that $K^- \leqq 0$. Applying the previous lemma to K^- yields $J_T(x_0, u) \geqq x_0^T K^- x_0$ for all u and $T \geqq 0$. □

Remark 4.5. In [17] it is suggested that the converse of the above lemma also holds, i.e., that finiteness of $V_f^+(x_0)$ for all x_0 implies that $\Gamma_- \neq \varnothing$. We were able neither to establish a proof nor to construct a counterexample to this assertion. We were, however, able to relate the condition $\Gamma_- \neq \varnothing$ to an equivalent one in terms of the quantities $J_T(x_0, u)$ in a slightly different way. Indeed, if (A, B) is controllable and $R > 0$ then the following equivalence can be proven:

$$\Gamma_- \neq \varnothing \Leftrightarrow \inf\{\liminf_{T \to \infty} J_T(x_0, u) \mid u \in L_{2,\mathrm{loc}}(\mathbb{R}^+)\} \text{ is finite for all } x_0 \in \mathbb{R}^n. \tag{4.3}$$

Note that if we could prove the above equivalence with $L_{2,\mathrm{loc}}(\mathbb{R}^+)$ replaced by $U(x_0)$ we would be done. Indeed, for $u \in U(x_0)$ we have $\liminf_{T \to \infty} J_T(x_0, u) = \lim_{T \to \infty} J_T(x_0, u) = J(x_0, u)$, so the infimum in (4.3) would then be equal to $V_f^+(x_0)$. We close this remark by concluding that finding tractable necessary and sufficient conditions for the finiteness of V_f^+ remains a difficult open problem (see also [18], [11], and [13]).

A final point we want to make here is that for the free-endpoint problem with indefinite cost, even if the optimal cost is finite for all initial conditions, it is not true in general that optimal controls *exist* for all initial conditions. We will illustrate this in the example below. It should therefore be clear that part of our problem is to formulate necessary and sufficient conditions for the existence of these optimal controls (as was also done in Theorem 4.1(iii)).

Example 4.6. Consider the controllable system $\dot{x} = -x + u$, $x(0) = x_0$ with indefinite cost functional

$$J(x_0, u) = \int_0^\infty -x(t)^2 + u(t)^2 \, dt,$$

that is, take $A = -1$, $B = 1$, $Q = -1$, $S = 0$, and $R = 1$. The corresponding ARE is given by $-2K - K^2 - 1 = 0$. Consequently, $K^- = K^+ = -1$. We claim that $V_f^+(x_0) = -x_0^2$. We will show this "from first principles." Let $u \in L_{2,\mathrm{loc}}(\mathbb{R}^+)$. For every $T \geqq 0$ we have

$$\begin{aligned}\int_0^T -x^2 + u^2 \, dt &= \int_0^T (x-u)^2 \, dt + 2\int_0^T x(-x+u)\, dt \\ &= \int_0^T (x-u)^2 \, dt + 2\int_0^T x\dot{x}\, dt = \int_0^T (x-u)^2\, dt + x^2(T) - x_0^2.\end{aligned}$$

Consequently, $J(x_0, u) \geqq -x_0^2$ for all $u \in U(x_0)$. On the other hand, for $\varepsilon > 0$ define $u = (1-\varepsilon)x$. Then $\dot{x} = -\varepsilon x$ and

$$J(x_0, u) = [(1-\varepsilon)^2 - 1]x_0^2 \int_0^\infty e^{-2\varepsilon t}\, dt = -x_0^2 + \frac{\varepsilon}{2} x_0^2.$$

It follows that $V_f^+(x_0) = \inf\{J(x_0, u) | u \in U(x_0)\} = -x_0^2$. Thus, we see that the optimal cost is finite (as could also be deduced from the fact that $K^- = -1 \leqq 0$). We claim, however, that *no optimal control exists*! Indeed, assume u^* is optimal. Let x^* be the corresponding trajectory. We have

$$-x_0^2 = J(x_0, u^*) = -x_0^2 + \lim_{T\to\infty} \left(\int_0^T (x^* - u^*)^2 \, dt + x^*(T)^2 \right).$$

From this it follows that $\int_0^\infty (x^* - u^*)^2 \, dt = 0$ and that, consequently, $u^* = x^*$. However, using this feedback control law yields $J(x_0, u^*) = 0$. If $x_0 \neq 0$ this yields a contradiction.

5. The free-endpoint regular LQ-problem with indefinite cost. In this section we will resolve the free-endpoint version of the regular LQ problem with indefinite cost functional. In the sequel, an important role will be played by the subspace

$$N := \langle \ker K^- | A^- \rangle \cap X^+(A^-). \tag{5.1}$$

By definition of A^- it is immediately clear that, in fact,

$$N = \langle \ker K^- | A - BR^{-1}S \rangle \cap X^+(A - BR^{-1}S). \tag{5.2}$$

Obviously, N is equal to the undetectable subspace of (K^-, A^-) with respect to the stability set $\mathbb{C}_g = \mathbb{C}^- \cup \mathbb{C}^0$. We also note that N is an A^--invariant subspace of $X^+(A^-)$. By Theorem 3.1, N corresponds to a real symmetric solution $\gamma(N)$ of the ARE. Let P_N be the projector onto N along $\Delta^{-1}N^\perp$. Then this solution $\gamma(N)$ is given by

$$K_f^+ := \gamma(N) = K^- P_N + K^+(I - P_N). \tag{5.3}$$

It will turn out that K_f^+, the solution of the ARE supported by the subspace N, is the bottleneck in the problem we want to resolve. We will show that the optimal cost for the free-endpoint problem is obtained from K_f^+ and that the optimal controls, if they exist, are given by the feedback control law $u = -R^{-1}(B^T K_f^+ + S)x$. Before stating the exact result we first give an intuitive argument as to exactly why the subspace N given by (5.1) is the "right" supporting subspace. The argument is as follows. First recall that if $\omega \geqq 0$, then the optimal cost for the free-endpoint problem is obtained from the smallest positive semidefinite solution of the ARE (see Theorem 4.2). Now, it can be shown that, again if $\omega \geqq 0$, $K = \gamma(V)$ is positive semidefinite if and only if $V \subset \ker K^-$ (see Theorem 6.2). Consequently, if $\omega \geqq 0$ then the optimal cost is obtained from the smallest solution $K = \gamma(V)$ of the ARE such that $V \subset \ker K^-$. Now, our choice to consider exactly the subspace N given by (5.1) is based on the guess that the latter statement is also valid if ω is indefinite. Note that K_f^+ is indeed the smallest solution of ARE for which its supporting subspace is contained in $\ker K^-$: if $K = \gamma(V)$ is such that $V \subset \ker K^-$ then, since V is A^--invariant, we must have $V \subset \langle \ker K^- | A^- \rangle$ (the latter being the largest A^--invariant subspace in $\ker K^-$). Also, $V \subset X^+(A^-)$. Thus, $V \subset N$. Then it follows from Theorem 3.2 that $K_f^+ \leqq K$. The following theorem is the main result of this paper.

THEOREM 5.1. *Let (A, B) be controllable and assume that $R > 0$. Then we have the following*:

(i) *$V_f^+(x_0)$ is finite for all $x_0 \in \mathbb{R}^n$ if the* ARE *has a negative semidefinite real symmetric solution* (*i.e.*, $\Gamma_- \neq \varnothing$).

(ii) *If $\Gamma_- \neq \varnothing$ then for all $x_0 \in \mathbb{R}^n$, $V_f^+(x_0) = x_0^T K_f^+ x_0$.*

(iii) *If $\Gamma_- \neq \varnothing$ then for all $x_0 \in \mathbb{R}^n$ there exists an optimal input u^* if and only if* $\ker \Delta \subset \ker K^-$.

(iv) *If $\Gamma_- \neq \varnothing$ and if* $\ker \Delta \subset \ker K^-$, *then for each $x_0 \in \mathbb{R}^n$ there is exactly one optimal input u^* and, moreover, this input is given by the feedback control law $u^* = -R^{-1}(B^T K_f^+ + S)x$.*

In the remainder of this section we will establish a proof of this theorem. In order to streamline this proof, we will state some of the most important ingredients as separate lemmas. In the first two lemmas, we will formulate some general structural properties of linear systems.

LEMMA 5.2. *Consider the system $\dot{x} = Ax + v$, $y = Cx$. Assume that (C, A) is observable. Let $v \in L_2(\mathbb{R}^+)$, $y \in L_\infty(\mathbb{R}^+)$. Then for every initial condition x_0 we have $x \in L_\infty(\mathbb{R}^+)$.*

Proof. Since (C, A) is observable there exists a matrix L such that $\sigma(A + LC) \subset \mathbb{C}^-$. Obviously, x satisfies the differential equation

$$\dot{x} = (A + LC)x - Ly + v, \qquad x(0) = x_0.$$

Using the variations of constants formula, together with some straightforward estimates, it is then easily verified that $x \in L_\infty(\mathbb{R}^+)$. □

Using the previous lemma we arrive at the following result that will be one of the main instruments in the proof of Theorem 5.1.

LEMMA 5.3. *Consider the system $\dot{x} = Ax + v$, $y = Cx$. Let $\mathbb{C}_g$ be a symmetric subset of $\mathbb{C}$. Assume that (C, A) is detectable with respect to $\mathbb{C}_g$. Let the state space $\mathbb{R}^n$ be decomposed into $\mathbb{R}^n = X_1 \oplus X_2$, where X_1 is A-invariant. In this decomposition, let $x = \binom{x_1}{x_2}$. Assume that $\sigma(A|X_1) \subset \mathbb{C}_g$ and $\sigma(A|\mathbb{R}^n/X_1) \subset \mathbb{C}_b$. Then for every initial condition x_0 we have*: *if $v \in L_2(\mathbb{R}^+)$ and $y \in L_\infty(\mathbb{R}^+)$ then $x_2 \in L_\infty(\mathbb{R}^+)$.*

Proof. We claim that, in fact, $X_1 = X_g(A)$. Indeed, the fact that $X_1 \subset X_g(A)$ is immediate. Denote $\sigma_0 := \sigma(A|X_g(A)/X_1)$. Then $\sigma_0 \subset \sigma(A|X_g(A)) \subset \mathbb{C}_g$. Also, $\sigma_0 \subset \sigma(A|\mathbb{R}^n/X_1) \subset \mathbb{C}_b$. This can only be the case if $\sigma_0 = \varnothing$ or, equivalently, if $X_1 = X_g(A)$.

By the fact that (C, A) is detectable with respect to $\mathbb{C}_g$ we may therefore conclude that $\langle \ker C|A\rangle \subset X_1$. Decompose $X_1 = X_{11} \oplus X_{12}$, with $X_{11} := \langle \ker C|A\rangle$ and X_{12} arbitrarily. Accordingly, let $x_1 = \binom{x_{11}}{x_{12}}$. We then have $\mathbb{R}^n = X_{11} \oplus X_{12} \oplus X_2$ with $x = (x_{11}^T, x_{12}^T, x_2^T)^T$. In this decomposition, let

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix}, \quad C = (0, C_2, C_3), \quad v = \begin{pmatrix} v_{11} \\ v_{12} \\ v_2 \end{pmatrix}.$$

Obviously, the system

$$\left((C_2, C_3), \begin{pmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{pmatrix} \right)$$

is observable. Moreover,

$$\begin{pmatrix} \dot{x}_{12} \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{pmatrix} \begin{pmatrix} x_{12} \\ x_2 \end{pmatrix} + \begin{pmatrix} v_{12} \\ v_2 \end{pmatrix}, \qquad y = (C_2, C_3) \begin{pmatrix} v_{12} \\ v_2 \end{pmatrix}.$$

It thus follows from Lemma 5.2 that $\binom{x_{12}}{x_2} \in L_\infty(\mathbb{R}^+)$, which of course implies that $x_2 \in L_\infty(\mathbb{R}^+)$. □

Another important instrument in the proof that we will establish is the following result.

LEMMA 5.4. *Consider the system $\dot{x} = Ax + Bu$, $x(0) = x_0$. Assume that (A, B) is controllable and $\sigma(A) \subset \mathbb{C}^- \cup \mathbb{C}^0$. Then for all $\varepsilon > 0$ there exists a control $u \in L_2(\mathbb{R}^+)$ such that $\int_0^\infty \|u(t)\|^2\, dt < \varepsilon$ and $x(x_0, u)(t) \to 0 (t \to \infty)$.*

Proof. For the given system consider the fixed-endpoint regular LQ problem

$$\inf \left\{ \int_0^\infty \|u(t)\|^2\, dt \,|\, u \in L_2(\mathbb{R}^+) \text{ and } x(x_0, u)(t) \to 0,\ t \to \infty \right\}.$$

It is well known (see also Theorem 4.1) that the above infimum is equal to $x_0^T K^+ x_0$, where K^+ is the maximal solution to the ARE: $A^T K + KA = KBB^T K$. We claim that $K^+ = 0$. Assume $K^+ \neq 0$. Since $K = 0$ is a solution to the ARE, we must have $0 \leqq K^+$. So, $K^+ \geqq 0$ and $K^+ \neq 0$. Consequently, there exists an orthogonal matrix S such that

$$SK^+S^T = \begin{pmatrix} K_1 & 0 \\ 0 & 0 \end{pmatrix},$$

with $K_1 > 0$. Denote $\bar{K} := SK^+S^T$, $\bar{A} := SAS^T$, $\bar{B} := SB$. Then we have $\bar{A}^T\bar{K} + \bar{K}\bar{A} = \bar{K}\bar{B}\bar{B}^T\bar{K}$. Decompose

$$\bar{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad \bar{B} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

It is easily seen that $A_{11}^T K_1 + K_1 A_{11} = K_1 B_1 B_1^T K_1$. Also, $K_1 A_{12} = 0$. Since $K_1 > 0$, this implies $A_{12} = 0$. Define $P := K_1^{-1}$. Then $P > 0$ and satisfies the Lyapunov equation $PA_{11}^T + A_{11}P = B_1 B_1^T$. Since (A_{11}, B_1) is controllable, this implies $\sigma(A_{11}) \subset \mathbb{C}^+$ (see, e.g., [21, Lemma 12.2]. This, however, contradicts the fact that $\sigma(A_{11}) \subset \sigma(\bar{A}) = \sigma(A) \subset \mathbb{C}^- \cup \mathbb{C}^0$. We conclude that the above infimum is zero. □

We have now collected the most important ingredients we need in the proof of our main theorem. In order to give this proof, we shall make a suitable direct sum decomposition of the state space. Let K_f^+ be the solution of the ARE (3.1) defined by (5.3). Denote $A_f^+ := A - BR^{-1}(B^T K_f^+ + S)$. By Theorem 3.1 we have

$$X^+(A_f^+) = N,$$
$$X^0(A_f^+) = \ker \Delta,$$
$$X^-(A_f^+) = X^-(A^+) \cap \Delta^{-1} N^\perp.$$

Define $X_1 := X^+(A_f^+)$, $X_2 := X^0(A_f^+)$, and $X_3 := X^-(A_f^+)$. Then $\mathbb{R}^n = X_1 \oplus X_2 \oplus X_3$. Since X_1 is A^--invariant and since X_2 is also A^--invariant ($\ker \Delta = X^0(A_K)$ for *all* $K \in \Gamma$) we have

$$A^- = \begin{pmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix}, \tag{5.4}$$

for given matrices A_{ij}. We also have $K_f^+ x = K^- x$ for all $x \in N$, and hence $A_f^+ | X_1 = A^- | X_1$. Also, since $\ker \Delta \subset \Delta^{-1} N^\perp$ and therefore $\ker \Delta \subset \ker P_N$, for all $x \in \ker \Delta$ we have $K_f^+ x = K^+ x = K^- x$. Hence $A_f^+ | X_2 = A^- | X_2$. Consequently,

$$A_f^+ = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A'_{33} \end{pmatrix}, \tag{5.5}$$

for a given matrix A'_{33}. Note that $\sigma(A_{11}) \subset \mathbb{C}^+$, $\sigma(A_{22}) \subset \mathbb{C}^0$ and $\sigma(A'_{33}) \subset \mathbb{C}^-$. Since $X_1 \subset \ker K^-$ and K^- is symmetric,

$$K^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & K_{22}^- & K_{23}^- \\ 0 & K_{23}^{-T} & K_{33}^- \end{pmatrix}. \tag{5.6}$$

Furthermore, we claim that Δ has the form

$$\Delta = \begin{pmatrix} \Delta_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta_{33} \end{pmatrix}.$$

Indeed, by Theorem 3.1 we have $X_2 \oplus X_3 = \Delta^{-1} X_1^\perp$ and therefore we must have $\Delta_{13} = 0$. The other zero blocks are caused by the fact that $X_2 = \ker \Delta$ and by the symmetry of Δ. Combining the representations for K^- and Δ, we find

$$K^+ = \begin{pmatrix} K_{11}^+ & 0 & 0 \\ 0 & K_{22}^+ & K_{23}^+ \\ 0 & K_{23}^{+T} & K_{33}^+ \end{pmatrix}$$

for given matrices K_{ij}^+ (note that, in fact, $K_{23}^+ = K_{23}^-$ and $K_{22}^+ = K_{22}^-$). Combining all this, we find that

$$K_f^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & K_{22}^- & K_{23}^- \\ 0 & K_{23}^{-T} & K_{33}^+ \end{pmatrix}. \tag{5.7}$$

We now proceed with the following lemma, which states that K_f^+ gives a lower bound for the optimal cost of the free-endpoint regular LQ problem.

LEMMA 5.5. *Assume that (A, B) is controllable, $R>0$, and $\Gamma_- \neq \emptyset$. For all $x_0 \in \mathbb{R}^n$ and for all $u \in U(x_0)$ we have*

$$J(x_0, u) \geqq x_0^T K_f^+ x_0 + \int_0^\infty \|u(t)+R^{-1}(B^T K_f^+ + S)x(t)\|_R^2 \, dt. \tag{5.8}$$

Here we have denoted $x(t) := x(x_0, u)(t)$.

Proof. Since $\Gamma_- \neq \emptyset$ we have $K^- \leqq 0$. Let $u \in U(x_0)$. It follows from Lemma 4.4 that $J(x_0, u)$ is either finite or equal to $+\infty$. Indeed, $J(x_0, u) = -\infty$ would imply $V_f^+(x_0) = -\infty$, which would contradict $\Gamma_- \neq \emptyset$. Of course, if $J(x_0, u) = +\infty$ then (5.8) holds trivially. Now assume that $J(x_0, u)$ is *finite*. By the fact that $K^- \leqq 0$ it follows from Lemma 4.3 that for all $T \geqq 0$

$$\int_0^T \|u(t)+R^{-1}(B^T K^- + S)x(t)\|_R^2 \, dt \leqq J_T(x_0, u) - x_0^T K^- x_0.$$

Denote $\nu(t) := u(t) + R^{-1}(B^T K^- + S)x(t)$. It then follows that $\int_0^\infty \|\nu(t)\|_R^2 \, dt < +\infty$, and hence that $\nu \in L_2(\mathbb{R}^+)$. Again using Lemma 4.3 and the fact that $-K^- \geqq 0$, we find that this implies $\lim_{T\to\infty} x^T(T) K^- x(T)$ exists (and is finite). Thus $K^- x$ must be bounded on $\mathbb{R}^+$. Denote $y(t) := K^- x(t)$. Since $\dot{x} = Ax + Bu$, we have that x, ν, and y are related by the equations

$$\dot{x} = A^- x + B\nu, \qquad y = K^- x.$$

Now let $\mathbb{R}^n$ be composed into $\mathbb{R}^n = X_1 \oplus X_2 \oplus X_3$ as introduced above. Write $K^- = (0, K_2^-, K_3^-)$, $B = (B_1^T, B_2^T, B_3^T)^T$, and $x = (x_1^T, x_2^T, x_3^T)^T$. Since $X_1 = N$ is the undetectable subspace (with respect to $\mathbb{C}^- \cup \mathbb{C}^0$) of (K^-, A^-), it is easily verified that the pair

$$\left((K_2^-, K_3^-), \begin{pmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{pmatrix} \right)$$

is detectable (with respect to $\mathbb{C}^- \cup \mathbb{C}^0$). Since $\sigma(A^-) \subset \mathbb{C}^+ \cup \mathbb{C}^0$ and since $X_2 = X^0(A^-)$, it can be verified that

$$\sigma\left(\begin{pmatrix} A_{11} & A_{13} \\ 0 & A_{33} \end{pmatrix} \right) \subset \mathbb{C}^+.$$

Hence, $\sigma(A_{22}) \subset \mathbb{C}^0$ and $\sigma(A_{33}) \subset \mathbb{C}^+$. Also, we have

$$\begin{pmatrix} \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} A_{22} & A_{23} \\ 0 & A_{33} \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} B_2 \\ B_3 \end{pmatrix} \nu, \qquad y = (K_2^-, K_3^-) \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}.$$

Since $\nu \in L_2(\mathbb{R}^+)$ and $y \in L_\infty(\mathbb{R}^+)$, by Lemma 5.3 (applied with $\mathbb{C}_g = \mathbb{C}^- \cup \mathbb{C}^0$) we have that $x_3 \in L_\infty(\mathbb{R}^+)$.

Again by applying Lemma 4.3, this time with $K = K_f^+$, we find that for all $T \geqq 0$

$$J_T(x_0, u) = \int_0^T \|u(t)+R^{-1}(B^T K_f^+ + S)x(t)\|_R^2 \, dt + x_0^T K_f^+ x_0 - x^T(T) K_f^+ x(T). \tag{5.9}$$

Denote $w(t) := u(t) + R^{-1}(B^T K_f^+ + S)x(t)$. Combining (5.6), (5.7), and (5.9), we obtain that for all $T \geqq 0$

$$J_T(x_0, u) = \int_0^T \|w(t)\|_R^2 \, dt + x_0^T K_f^+ x_0 - x_3^T(T) \Delta_{33} x_3(T) - x^T(T) K^- x(T). \tag{5.10}$$

Recall that $\lim_{T\to\infty} J_T(x_0, u)$ was assumed to be finite. Thus, $J_T(x_0, u)$ is a bounded function of T. Since also $x_3(T)$ and $x^T(T) K^- x(T)$ are bounded functions of T, (5.10) implies that $\int_0^\infty \|w(t)\|_R^2 \, dt < \infty$. It follows that $w \in L_2(\mathbb{R}^+)$.

We again consider (5.10). Since now $J_T(x_0, u)$, $\int_0^T \|w(t)\|_R^2\, dt$ and $x^T(T)K^- x(T)$ *converge* for $T \to \infty$, it follows that $\lim_{T\to\infty} x_3^T(T)\Delta_{33}x_3(T)$ exists. Since $\Delta_{33} > 0$ this implies that $\|x_3(T)\|$ converges as $T \to \infty$. Now, since $\dot{x} = Ax + Bu$, the variables x and w are related via $\dot{x} = A_f^+ x + Bw$, and hence (see 5.5) $\dot{x}_3 = A'_{33}x_3 + B_3 w$. Since $w \in L_2(\mathbb{R}^+)$ and $\sigma(A'_{33}) \subset \mathbb{C}^-$ we have that $x_3 \in L_2(\mathbb{R}^+)$. A fortiori, since $\|x_3(t)\|$ converges as $t \to \infty$, this yields $\lim_{t\to\infty} x_3(t) = 0$. Using this, and the fact that $-K^- \geqq 0$, it then follows from (5.10) that (5.8) holds. □

Our next lemma states that, by choosing the control properly, the difference between K_f^+ and the value of the cost functional can be made arbitrarily small.

LEMMA 5.6. *Assume that* (A, B) *is controllable,* $R > 0$, *and* $\Gamma \neq \emptyset$. *Then for all* $x_0 \in \mathbb{R}^n$ *and for all* $\varepsilon > 0$ *there exists an input* $u \in U(x_0)$ *such that* $J(x_0, u) \leqq x_0^T K_f^+ x_0 + \varepsilon$.

Proof. Again, let $\mathbb{R}^n$ be decomposed as above. It follows from (5.7) and (5.9) that for all $u \in L_{2,\mathrm{loc}}(\mathbb{R}^+)$ and for all $T \geqq 0$

$$(5.11)\quad J_T(x_0, u) = \int_0^T \|w(t)\|_R^2\, dt + x_0^T K_f^+ x_0 - (x_2^T(T), x_3^T(T))\begin{pmatrix} K_{22}^- & K_{23}^- \\ K_{23}^{-T} & K_{33}^+ \end{pmatrix}\begin{pmatrix} x_2(T) \\ x_3(T) \end{pmatrix}.$$

Here, $w := u + R^{-1}(B^T K_f^+ + S)x$. Since $\dot{x} = Ax + Bu$, the variables x and w are related by $\dot{x} = A_f^+ x + Bw$, and hence (see (5.5))

$$\begin{pmatrix} \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} A_{22} & 0 \\ 0 & A'_{33} \end{pmatrix}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} B_2 \\ B_3 \end{pmatrix} w.$$

Note that $\sigma(A_{22}) \subset \mathbb{C}^0$, $\sigma(A'_{33}) \subset \mathbb{C}^-$ and that this system is controllable. Now let $\varepsilon > 0$. It follows from Lemma 5.4 that there exists a control $w \in L_2(\mathbb{R}^+)$ such that $\int_0^\infty \|w(t)\|_R^2\, dt < \varepsilon$ and such that $x_2(T) \to 0$ and $x_3(T) \to 0$ as $T \to \infty$. Define $u := -R^{-1}(B^T K_f^+ + S)x + w$. Then we have

$$J(x_0, u) = \lim_{T\to\infty} J_T(x_0, u) = \int_0^\infty \|w(t)\|_R^2\, dt + x_0^T K_f^+ x_0 \leqq \varepsilon + x_0^T K_f^+ x_0. \qquad \square$$

We will now prove our main theorem.

Proof of Theorem 5.1. (i) This proof was already stated separately in Lemma 4.4.

(ii) Lemma 5.5 yields $J(x_0, u) \geqq x_0^T K_f^+ x_0$ for all $u \in U(x_0)$. Together with Lemma 5.6 this implies $V_f^+(x_0) = x_0^T K_f^+ x_0$ for all x_0.

(iii) Assume $\Gamma_- \neq \emptyset$. ($\Rightarrow$) Assume that for all x_0 there exists a control $u^* \in U(x_0)$ such that $J(x_0, u^*) = V_f^+(x_0) = x_0^T K_f^+ x_0$. Let $x_0 \in \mathbb{R}^n$ be arbitrary and let u^* be the corresponding optimal control. Denote $x^* := x(x_0, u^*)$. By Lemma 5.5

$$x_0^T K_f^+ x_0 = J(x_0, u^*) \geqq x_0^T K_f^+ x_0 + \int_0^\infty \|u^*(t) + R^{-1}(B^T K_f^+ + S)x^*(t)\|_R^2\, dt.$$

It follows that u^* must be given by the feedback control law $u^* = -R^{-1}(B^T K_f^+ + S)x^*$. This implies that x^* satisfies the equation $\dot{x}^* = A_f^+ x^*$. In terms of the decomposition introduced above, this of course yields $\dot{x}_2^* = A_{22}x_2^*$ and $\dot{x}_3^* = A'_{33}x_3^*$ (see 5.5). Since $\sigma(A'_{33}) \subset \mathbb{C}^-$ we must have $x_3^*(t) \to 0 (t \to \infty)$. By (5.10)

$$J_T(x_0, u^*) = x_0^T K_f^+ x_0 - x_3^{*T}(T)\Delta_{33}x_3^*(T) - x^{*T}(T)K^- x^*(T).$$

By the fact that $J_T(x_0, u^*) \to x_0^T K_f^+ x_0$ we obtain that $x^{*T}(T)K^- x^*(T) \to 0$ $(T \to \infty)$. Since K^- is semidefinite, a fortiori this implies $K^- x^*(T) \to 0$ $(T \to \infty)$. Using (5.6) this yields

$$K_{22}^- x_2^*(T) + K_{23}^- x_3^*(T) \to 0 \qquad (T \to \infty).$$

Since $x_3^*(T)\to 0$ $(T\to\infty)$ the latter implies $K_{22}^- x_2^*(T)\to 0$ $(T\to\infty)$ or, equivalently, $K_{22}^- \exp(A_{22}T)x_2(0)\to 0$ $(T\to\infty)$. Now, $x_2(0)$ was completely arbitrary and therefore we find that

$$K_{22}^- e^{A_{22}T}\to 0 \qquad (T\to\infty).$$

Consequently, $K_{22}^-(Is-A_{22})^{-1}$ has all its poles in $\mathbb{C}^-$. On the other hand, however, since $\sigma(A_{22})\subset\mathbb{C}^0$, it has all its poles in $\mathbb{C}^0$. Thus, $K_{22}^-(Is-A_{22})^{-1}=0$, and hence $K_{22}^-=0$. Since K^- is semidefinite this implies $K_{23}^-=0$. It follows that $\ker\Delta=X_2\subset\ker K^-$.

($\Leftarrow$) Conversely, assume $\ker\Delta\subset\ker K^-$. Then $K_{22}^-=0$ and $K_{23}^-=0$. Define $u=-R^{-1}(B^TK_f^+ + S)x$. We claim that this feedback law yields an optimal u. Indeed, by (5.11)

$$J_T(x_0,u)=x_0^TK_f^+x_0-x_3^T(T)K_{33}^+x_3(T).$$

Moreover, $\dot{x}_3=A_{33}'x_3$. Since $\sigma(A_{33}')\subset\mathbb{C}^-$ we have $x_3(T)\to 0$ $(T\to\infty)$. Thus $J(x_0,u)=x_0^TK_f^+x_0=V_f^+(x_0)$, so u is optimal.

(iv) The fact that $u^*=-R^{-1}(B^TK_f^+ + S)x^*$ is *unique* was already proven in (iii) ($\Rightarrow$). This concludes the proof of our theorem. □

Remark 5.7. At this point we would like to mention that, in addition to the option we have chosen in § 2, there is still another very natural and appealing way to formulate the regular LQ problem. Instead of restricting the class of controls to $U(x_0)$ in order to guarantee that the indefinite integrals in (2.2) are well-defined, it is also possible to choose $L_{2,\mathrm{loc}}(\mathbb{R}^+)$ for the class of admissible controls and to consider the following cost functional:

$$\tilde{J}(x_0,u):=\limsup_{T\to\infty} J_T(x_0,u).$$

Obviously, on the subclass $U(x_0)\subset L_{2,\mathrm{loc}}(\mathbb{R}^+)$ the functionals $\tilde{J}(x_0,\cdot)$ and $J(x_0,\cdot)$ coincide. Corresponding to this choice of cost functional, we can now consider the following version of the free-endpoint regular LQ problem:

$$\tilde{V}_f^+(x_0):=\inf\{\tilde{J}(x_0,u)\,|\,u\in L_{2,\mathrm{loc}}(\mathbb{R}^+)\}.$$

As it turns out, we can develop around this version of the problem a theory completely parallel to the one we developed in this section. In fact, Theorem 5.1 remains valid if in its statement we replace V_f^+ by $\tilde{V}_f^+$! In particular, both problems yield the same optimal controls u^*. Consequently, if u^* is optimal for the problem with functional $\tilde{J}(x_0,\cdot)$, then in fact $u^*\in U(x_0)$ and $\tilde{V}_f^+(x_0)=\tilde{J}(x_0,u^*)=\lim_{T\to\infty}J_T(x_0,u^*)$. Similar remarks hold for the fixed-endpoint problem.

6. Comparison and special cases. In this section we will discuss some questions that arise if we compare the optimal costs and optimal closed loop systems resulting from the free-endpoint and fixed-endpoint problem, respectively. In particular, we will establish conditions under which the respective optimal costs are the same. Also, conditions will be found under which the free-endpoint optimal closed loop system is asymptotically stable. Finally, we will show how our general results can be specialized to reobtain the most important results on the free-endpoint regular LQ problem with *positive semidefinite* cost functional. First, we have the following theorem.

THEOREM 6.1. *Assume that* (A,B) *is controllable,* $R>0$, *and* $\Gamma\neq\emptyset$. *Then we have the following*:

(i) $K_f^+=K^+$ *if and only if the pair* $(K^-, A-BR^{-1}S)$ *is detectable with respect to the stability set* $\mathbb{C}^-\cup\mathbb{C}^0$.

(ii) $\sigma(A_f^+)\subset\mathbb{C}^-$ *if and only if the pair* $(K^-, A-BR^-S)$ *is detectable with respect to* $\mathbb{C}^-$ *and* $\Delta>0$.

Proof. (i) By (5.2), N is equal to the undetectable subspace of $(K^-, A-BR^{-1}S)$ with respect to $\mathbb{C}^-\cup\mathbb{C}^0$. Since K^+ is supported by the zero subspace, by Theorem 3.1 we have $K_f^+=K^+$ if and only if $N=0$.

(ii) ($\Leftarrow$) Detectability with respect to $\mathbb{C}^-$ implies detectability with respect to $\mathbb{C}^-\cup\mathbb{C}^0$. Hence $K_f^+=K^+$ and $A_f^+=A^+$. By [17, Thm. 5] $\Delta>0$ if and only if $\sigma(A^+)\subset\mathbb{C}^-$.

($\Rightarrow$) Conversely, assume $\sigma(A_f^+)\subset\mathbb{C}^-$. By [17, Thm. 5] there is exactly one $K\in\Gamma$, namely $K=K^+$, such that $\sigma(A_K)\subset\mathbb{C}^-\cup\mathbb{C}^0$. Hence $K_f^+=K^+$, $A_f^+=A^+$. Consequently, $\Delta>0$. Also, from (i) we obtain that the pair $(K^-, A-BR^{-1}S)$ is detectable with respect to $\mathbb{C}^-\cup\mathbb{C}^0$. Since $\Delta>0$, $\sigma(A^-)\subset\mathbb{C}^+$. Hence $X^0(A^-)=0$ so $(K^-, A-BR^{-1}S)$ is in fact detectable with respect to $\mathbb{C}^-$. □

We will now discuss how our results can be specialized to rederive some important "classical" results on the special case that the quadratic form ω is positive semidefinite. We have the following characterization of the positive semidefinite solutions of the ARE.

THEOREM 6.2. *Assume that* (A, B) *is controllable,* $R>0$, $\Gamma_-\neq\varnothing$, *and* $\Gamma_+\neq\varnothing$. *Let* $K\in\Gamma$ *be supported by* V. *Then* $K\in\Gamma_+$ *if and only if* $V\subset\ker K^-$.

Proof. By Theorem 3.1 we have $V\oplus\Delta^{-1}V^\perp=\mathbb{R}^n$.

($\Leftarrow$) Assume that $V\subset\ker K^-$. Then $\Delta^{-1}V^\perp=\{x\in\mathbb{R}^n \mid y^TK^+x=0, \text{for all } y\in V\}$ and $K=K^+(I-P_V)$. Let $x\in\mathbb{R}^n$, $x=x_1+x_2$ with $x_1\in V$ and $x_2\in\Delta^{-1}V^\perp$. It is easily seen that $x^TKx=x_2^TK^+x_2$. Since $\Gamma_+\neq\varnothing$ we have $K^+\geqq 0$. It follows that $K\geqq 0$.

($\Leftarrow$) Conversely, if $K\geqq 0$ then for all $x\in V$ we have

$$0\leqq x^TKx=x^T(K^-P_V+K^+(I-P_V))x=x^TK^-x.$$

Since $\Gamma_-\neq\varnothing$ we have $K^-\leqq 0$. It follows that $x^TK^-x=0$, and hence that $x\in\ker K^-$. □

Our next result states that, under the assumption that $\Gamma_-\neq\varnothing$, if the ARE has positive semidefinite solutions at all, then it has a smallest positive semidefinite solution and this solution is equal to the one supported by N.

THEOREM 6.3. *Assume that* (A, B) *is controllable,* $R>0$, *and* $\Gamma_-\neq\varnothing$. *Then the following hold: if* $\Gamma_+\neq\varnothing$ *then* (i) $K_f^+\in\Gamma_+$ *and* (ii) $K\in\Gamma_+$ *implies* $K_f^+\leqq K$.

Proof. Since $N\subset\ker K^-$ it follows from Theorem 6.2 that $K_f^+\in\Gamma_+$. Now assume $K\in\Gamma_+$ and K is supported by the A^--invariant subspace $V\subset X^+(A^-)$. Since $K\in\Gamma_+$ we have $V\subset\ker K^-$. Hence $V\subset\langle\ker K^-|A^-\rangle$ (the latter is the largest A^--invariant subspace in $\ker K^-$; see [21]). It follows that $V\subset N$. But then, by Theorem 3.2, $K_f^+\leqq K$. □

From the above we deduce the following remarkable fact. Consider the free-endpoint regular LQ problem with *indefinite* cost functional. Let (A, B) be controllable. We already saw that the optimal cost is finite if we have $\Gamma_-\neq\varnothing$. Assume this to be the case. Then Theorem 6.3 states that *if the* ARE *has at least one positive semidefinite solution, then the optimal cost is given by the smallest of these solutions*! The case that the cost functional is positive semidefinite, i.e., $\omega(x, u)\geqq 0$, for all (x, u), is in fact a special case of this general principle. Indeed, if (A, B) is controllable and if $\omega\geqq 0$ then $\Gamma_+\neq\varnothing$ (see [5]). Moreover, applying the latter to the controllable system $(-A, -B)$ and the same form $\omega\geqq 0$, we can also see that $\Gamma_-\neq\varnothing$. Thus we have reobtained Theorem 4.2(i).

Our next result shows that the fact that for the case $\omega\geqq 0$ optimal controls exist for all initial conditions is also a special case of a more general principle.

PROPOSITION 6.4. *Assume* (A, B) *is controllable,* $R>0$, $\Gamma_-\neq\varnothing$, *and* $\Gamma_+\neq\varnothing$. *Then* $\ker\Delta\subset\ker K^-$.

Proof. $\Gamma_- \neq \emptyset$ is equivalent to $K^- \leqq 0$ and $\Gamma_+ \neq \emptyset$ is equivalent to $K^+ \geqq 0$. Assume $x \in \ker \Delta$. Then $0 \leqq x^T K^+ x = x^T K^- x \leqq 0$. Thus $x^T K^- x = 0$, and hence $K^- x = 0$. □

By combining this with the above remarks and by applying Theorem 5.1(iii) and (iv) we reobtain Theorem 4.2(ii).

To conclude this section, we will briefly discuss what statements can be obtained from Theorem 6.1 for the case that our cost functional is positive semidefinite. In the rest of this section, assume that $\omega(x, u) \geqq 0$ for all (x, u). We claim that in this case

$$N = \langle \ker (Q - S^T R^{-1} S) | A - BR^{-1}S \rangle \cap X^+(A - BR^{-1}S). \tag{6.1}$$

First we claim that $\ker K^-$ is $(A - BR^{-1}S)$-invariant. Indeed, if $\omega \geqq 0$ then $Q - S^T R^{-1} S \geqq 0$. Also it is straightforward to verify that

$$(A - BR^{-1}S)^T K^- + K^-(A - BR^{-1}S) + Q - S^T R^{-1} S - K^- BR^{-1} B^T K = 0. \tag{6.2}$$

Let $x_0 \in \ker K^-$. Then from (6.2), $x_0^T(Q - S^T R^{-1} S)x_0 = 0$, and hence $(Q - S^T R^{-1} S)x_0 = 0$. Thus, again from (6.2), $K^-(A - BR^{-1}S)x_0 = 0$ so $(A - BR^{-1}S)x_0 \in \ker K^-$. It follows that $\langle \ker K^- | A - BR^{-1}S \rangle = \ker K^-$. Now, by using the interpretation of K^- as the optimal cost for a fixed-endpoint LQ problem in "reversed time" (see [21, Thm. 7]) it can be proved that

$$\begin{aligned} \ker K^- = \langle \ker (Q - S^T R^{-1} S) | A - BR^{-1}S \rangle \\ \cap (X^+(A - BR^{-1}S) \oplus X^0(A - BR^{-1}S)). \end{aligned} \tag{6.3}$$

Thus (6.1) follows immediately from (5.2). We have now shown that if $\omega \geqq 0$, then K_f^+ is in fact supported by the undetectable subspace of the pair $(Q - S^T R^{-1} S, A - BR^{-1}S)$ with respect to $\mathbb{C}^- \cup \mathbb{C}^0$. (See also [3, Thm. 1].) By applying Theorem 6.1(i) we may then conclude that $K_f^+ = K^+$ if and only if $(Q - S^T R^{-1} S, A - BR^{-1}S)$ is detectable with respect to $\mathbb{C}^- \cup \mathbb{C}^0$ (see also [12, Cor., p. 356]).

Finally, we will re-establish the well-known fact that $\sigma(A_f^+) \subset \mathbb{C}^-$ if and only if $(Q - S^T R^{-1} S, A - BR^{-1}S)$ is detectable with respect to $\mathbb{C}^-$ (see [6], [20], and [12]). Assume that $\omega \geqq 0$. We claim that if $(K^-, A - BR^{-1}S)$ is detectable with respect to $\mathbb{C}^-$ then $\Delta > 0$. Indeed, if $(K^-, A - BR^{-1}S)$ is detectable with respect to $\mathbb{C}^-$ then (K^-, A^-) is detectable with respect to $\mathbb{C}^-$. The latter is equivalent to

$$\langle \ker K^- | A^- \rangle \cap (X^+(A^-) \oplus X^0(A^-)) = 0. \tag{6.4}$$

By Theorem 3.1, $X^0(A^-) = \ker \Delta$. Also, since $\omega \geqq 0$, $\ker \Delta \subset \ker K^-$. Hence, by (6.4), $\ker \Delta + (\langle \ker K^- | A^- \rangle \cap X^+(A^-)) = 0$, whence $\ker \Delta = 0$. It follows that $\Delta > 0$. We may now conclude from Theorem 6.1(ii) that $\sigma(A_f^+) \subset \mathbb{C}^-$ if and only if the pair $(K^-, A - BR^{-1}S)$ is detectable with respect to $\mathbb{C}^-$. From the fact that $\ker K^-$ is $(A - BR^{-1}S)$-invariant and from (6.3), the latter condition is, however, equivalent to the statement that the pair $(Q - S^T R^{-1} S, A - BR^{-1}S)$ is detectable with respect to $\mathbb{C}^-$.

7. Concluding remarks. In this paper we have studied just one of the many open basic questions that still exist in the context of linear quadratic optimal control. To name but a few of these open problems, we mention, for example, the question about the relationship between the *finite*-horizon free-endpoint problem and the infinite-horizon free-endpoint problem. It is well known that if the cost functional is positive semidefinite, then the finite-horizon optimal cost converges to the infinite-horizon optimal cost [1], [2], [9]. It would be interesting to investigate whether this is also true for the indefinite case. Another open problem is the *singular* LQ problem with indefinite cost functional, that is, the problem studied here without the assumption that R is positive definite. Recently [19] this problem was treated for the case that the cost-functional is positive semidefinite. However, for both the free-endpoint case as well as the fixed-endpoint case, the indefinite version of this problem still remains to be solved.

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